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Phil. Trans. R. Soc. Lond. A 1955 **248**, 73-96
doi: 10.1098/rsta.1955.0010

ON THE PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS AND INDEFINITE TERNARY QUADRATIC FORMS

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(*Communicated by K. Mahler, F.R.S.—Received* 4 *October* 1954—
Revised 3 *January* 1955)

CONTENTS

Isolation theorems for the minima of factorizable homogeneous ternary cubic forms and of indefinite ternary quadratic forms of a new strong type are proved. The problems whether there exist such forms with positive minima other than multiples of forms with integer coefficients are shown to be equivalent to problems in the geometry of numbers of a superficially different type. A contribution is made to the study of the problem whether there exist real ϕ, ψ such that $x|\phi x-y|\,|\psi x-z|$ has a positive lower bound for all integers $x>0, y, z$. The methods used have wide validity.

Notation

Matrices are denoted by Gothic capitals $\mathfrak{D}$, $\mathfrak{I}$, $\mathfrak{T}$, etc., where $\mathfrak{I}$ is the unit matrix.

Lattices are denoted by Λ, M and their (common) determinant by Δ.

Regions of space are denoted by script capitals $\mathscr{R}$, $\mathscr{S}$.

Numbers and functions are denoted by small Greek or large or small Latin letters indifferently. We have endeavoured to retain conventional notation as far as possible.

Co-ordinate systems in three-dimensional space are denoted interchangeably by (x, y, z) or (x_1, x_2, x_3) according to convenience.

1. Introduction

Suppose that $f(x_1, \ldots, x_n)$ is an algebraic form† of some specific type—a quadratic or a product of linear forms, for example. For any $\epsilon > 0$ we define an ϵ-**neighbourhood** of f as the set of forms f^* which are of the same type as f and whose coefficients lie within ϵ of the corresponding coefficients of f. Any set which contains some ϵ-neighbourhood will be called a **neighbourhood**. The formalization of these concepts, which are obviously closely akin to the definition of Mahler (1946) for lattices, presents no difficulty; but there are two

† The term 'form' is taken throughout this paper to imply homogeneity.

points which it is important to note. We have a wide choice of possible representations of f; thus the representations

$$6x^2+5xy-6y^2, \quad (2x+3y)(3x-2y), \quad 6(x+\tfrac{3}{2}y)(x-\tfrac{2}{3}y),$$

where in the third case only 6, $\frac{3}{2}$ and $\frac{2}{3}$ are regarded as coefficients which one is permitted to vary, lead to the same set of neighbourhoods of f. Again, the neighbourhoods of xyz as a product of three linear forms, for example, are simply the intersection of its neighbourhoods as a ternary cubic form with the set of all products of three linear forms.

An important part in the geometry of numbers is played by the so-called local isolation theorems, of which the following is a typical example due to C. A. Rogers (unpublished).

THEOREM 1. *Let $f(x,y)$ be an indefinite binary quadratic form with integer coefficients which does not represent zero, so that*†

$$m = \min'|f(x,y)|>0.$$

Suppose that f takes both the values $+m$ and $-m$. Then there is a neighbourhood of f and an $m'<m$ such that, for every f^* in the neighbourhood which is not a multiple of f,

$$\min'|f^*(x,y)|<m'.$$

The first purpose of this paper is to show that a surprisingly stronger result holds for the product of three linear forms in three variables:

THEOREM 2. *Let $f(x,y,z) = L_1L_2L_3$ be the product of three real linear forms which represent zero only trivially,‡ and suppose that f has integer coefficients. Let (δ_1, δ_2) be any open interval however small. Then there is a neighbourhood of f such that all forms f^* in the neighbourhood which are not multiples of f itself take some value in the interval (δ_1, δ_2).*

In particular, to any given $\delta>0$ we can choose a neighbourhood in which

$$\min'|f^*(x,y,z)|<\delta.$$

Moreover, there is nothing in theorem 2 analogous to the special requirement in theorem 1 that f should take both the values $+m$ and $-m$. We have stated, and shall prove, theorem 2 for neighbourhoods in the set of products of three linear forms. It continues to hold for neighbourhoods in the set of all ternary cubic forms. The proof, by the methods of this paper, involves no point of real difficulty; we do not, however, give it, since it involves a tedious division into cases and we cannot conceive that the result will ever be of value to anyone.

The neighbourhood of f is obtained by making small variations of the coefficients in L_1, L_2, L_3. If we are allowed to vary only two of the forms we obtain the stronger result:

THEOREM 3. *Let L_1, L_2, L_3 satisfy the conditions of theorem 2, and let L_2^*, L_3^* be any real linear forms such that $L_2^*L_3^*$ is not a numerical multiple of L_2L_3. Then the set of values taken by $L_1L_2^*L_3^*$ is everywhere dense in $(-\infty, \infty)$.*

† We use 'min' to indicate a greatest lower bound, and are not concerned whether or not it is attained. For reasons of typography, we use 'min′' to denote a min taken over all integer values of the variables not all zero.

‡ I.e. only when $x=y=z=0$, in accordance with accepted terminology.

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It is trivial that if f or a multiple of it satisfies the conditions of theorem 2, then

$$\min' |f(x,y,z)| > 0.$$

It is an unsolved problem whether these are the only products of three linear forms with this property. The second object of this paper is to prove the equivalence of this problem with another superficially quite different—though equally intractable. We state

Hypothesis A. *There exist real linear forms L_1, L_2, L_3 in x, y, z such that $L_1L_2L_3$ is not a multiple of a form with integer coefficients, for which* $\min' |L_1L_2L_3| = 1$.

Hypothesis B. *There exist real linear forms M_1, M_2, M_3 in x, y, z such that*

$$\min' \min\{|M_1M_2M_3|, |M_1M_2(M_2+M_3)|\} = 1,$$

where min′ *is as usual and* min *is over the two numbers given.*

Hypothesis C. *There exist real ϕ, ψ such that*

$$\min |x(\phi x - y)(\psi x - z)| > 0,$$

where the min *is taken over all integers $x \neq 0, y, z$.*

We have for convenience stated all three of these in an affirmative form; we tend rather to believe, however, that they are all false. Certainly the most natural way of trying to satisfy A, by analogy with the binary case, would be to take L_1, L_2, L_3 as suitably chosen non-conjugate forms in the same totally real cubic field; and we know from theorem 3 that this cannot be effective. Hypothesis C represents a classical problem of Littlewood's, which he expressed in the equivalent form

$$\varliminf_{n\to\infty} n\,|\sin \pi\phi n \sin \pi\psi n| > 0.$$

It is well known that

$$\varliminf_{n\to\infty} n\,|\sin \pi\phi n| = 0$$

for almost all ϕ, which would tend to suggest that C is false; on the other hand, for any given $\epsilon > 0$,

$$\lim_{n\to\infty} n^{1+\epsilon}\,|\sin \pi\phi n \sin \pi\psi n| = \infty$$

for almost all pairs ϕ, ψ. We are indebted to Professor Littlewood, to whom this result is due, for permission to publish his proof. It will be found in appendix A.

Theorem 4. *Hypotheses* A *and* B *are equivalent. Moreover, if they are both true the lower bound of the determinant of forms L_1, L_2, L_3 satisfying* A *is the same as that of the determinant of forms M_1, M_2, M_3 satisfying* B.

Theorem 5. *If* A *is false then for any D_0 however large there are only a finite number of inequivalent sets of forms L_1, L_2, L_3 with determinant $\leqslant D_0$ such that*

$$\min' |L_1L_2L_3| = 1.$$

Here two sets of forms are considered equivalent if the corresponding products $L_1L_2L_3$ can be transformed one into the other by an integral unimodular transformation on x, y, z.

Theorem 6. C *implies both* A *and* B.

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THEOREM 7. *If C is true, then ϕ and ψ cannot be two elements of the same cubic field.*

With certain modifications, our methods may be expected to apply to all forms whose group of automorphisms is sufficiently large. We therefore prove the analogues† of theorems 2, 4 and 5 for indefinite ternary quadratic forms. The proofs do, however, involve considerable extra difficulties, principally because the group of automorphisms of the forms is no longer commutative. For binary quadratic forms the analogue of theorems 2 and 8 is clearly false. On the other hand, the argument leading to theorems 3 and 9 remains valid. We know from the theory of the Markoff chain that the analogue of hypothesis A is true and that the lower bound of the determinant of admissible forms L_1, L_2 is 3. We deduce that the star body

$$\min\{|XY|, |X(X+Y)|\} \leqslant 1,$$

in the plane, is of finite type and has critical determinant 3. It might be interesting to have a direct proof of this by the methods of the geometry of numbers.

It is perhaps worth remarking that if A and B and the corresponding hypotheses D and E below for indefinite ternary quadratic forms are false, then theorems 5 and 10 show that the chains of minima obtained by Davenport (1943) for ternary cubics and by Venkov (1945) and Oppenheim (1953) for indefinite ternary quadratics, may be carried arbitrarily far at the expense of a correspondingly great but strictly finite amount of computation.

We remark finally that hypothesis A would follow if there existed homogeneous ternary linear forms N_1, N_2, N_3 such that $\min'|N_1(N_2N_3+N_1^2)|=1$, as can be proved by our methods. The problem of the existence of such forms N_1, N_2, N_3 has been raised by Davenport & Rogers (1949).

Professor Davenport has done much to render this account intelligible.

2. PRELIMINARIES TO PROOF OF THEOREM 2

The proof of theorem 2 is based on the following variant of Kronecker's theorem, the relevance of which will soon be apparent.

LEMMA 1. *Let α, β, γ, δ be constants with $\alpha\delta-\beta\gamma\neq 0$. Suppose that α/β is irrational. Then to every $\tau>0$ there is a $\sigma=\sigma(\tau,\alpha,\beta,\gamma,\delta)$ with the following property:*

For any λ there are integers m, n such that

$$|m\alpha+n\beta-\lambda|<\tau, \quad |m\gamma+n\delta|\leqslant\sigma.$$

It is not difficult to deduce this from Kronecker's theorem but we give an independent proof.

By Minkowski's linear forms theorem and the fact that α/β is irrational there are integral m, n such that $m\alpha+n\beta$ is arbitrarily small but non-zero. Let

$$0<|m_1\alpha+n_1\beta|<\tau, \quad 0<|m_2\alpha+n_2\beta|<\tau,$$

$$m_1n_2-m_2n_1\neq 0,$$

and put
$$X_j=m_j\alpha+n_j\beta, \quad Y_j=m_j\gamma+n_j\delta \quad (j=1,2),$$

so
$$|X_j|<\tau \quad (j=1,2), \qquad X_1Y_2-X_2Y_1\neq 0.$$

† Theorems 8, 9 and 10, stated respectively in §§ 9, 10 and 11. We are unable even to state any analogue to theorem 3 or hypothesis C.

Let u, v be the solutions of

$$uX_1+vX_2=\lambda, \quad uY_1+vY_2=0,$$

and choose integers a, b such that

$$|a-u|\leqslant\tfrac{1}{2}, \quad |b-v|\leqslant\tfrac{1}{2}.$$

Then
$$|aX_1+bX_2-\lambda|\leqslant\tfrac{1}{2}(|X_1|+|X_2|)<\tau$$
and
$$|aY_1+bY_2|=|(a-u)Y_1+(b-v)Y_2|\leqslant\sigma,$$
where
$$\sigma=\tfrac{1}{2}(|Y_1|+|Y_2|).$$

Since aX_1+bX_2, aY_1+bY_2 are respectively the values taken by $m\alpha+n\beta$, $m\gamma+n\delta$ for

$$m=am_1+bm_2, \quad n=an_1+bn_2,$$

this proves the lemma.

We now consider theorem 2. It is known (Bachmann 1923) that if $L_1L_2L_3$ has integer coefficients and does not represent zero, then there are constants λ_1, λ_2, λ_3 such that $\lambda_1\lambda_2\lambda_3$ is integral and

$$\lambda_jL_j=\alpha_jx+\beta_jy+\gamma_jz \quad (j=1,2,3),$$

where α_1, β_1, γ_1 are linearly independent integers of a totally real cubic field K_1 and $\alpha_2, \beta_2, \gamma_2, \alpha_3, \beta_3, \gamma_3$ are their respective conjugates in the conjugate fields K_2, K_3. It is therefore enough to prove theorem 2 in the special case

$$L_j=\alpha_jx+\beta_jy+\gamma_jz.$$

We may now take L_1, L_2, L_3 as our new variables, writing

$$f^*=(1+\epsilon_0)L_1^*L_2^*L_3^*,$$

with

$$L_1^*=L_1+\epsilon_{12}L_2+\epsilon_{13}L_3,$$
$$L_2^*=\epsilon_{21}L_1+L_2+\epsilon_{23}L_3,$$
$$L_3^*=\epsilon_{31}L_1+\epsilon_{32}L_2+L_3.$$

As may readily be verified, the neighbourhoods of f may be given by bounds on ϵ_0 and the six ϵ_{ij}. We note that f^* is a multiple of f if and only if all the ϵ_{ij} vanish.

It follows from the theory of units of algebraic number fields (Bachmann 1923), that there are two independent units η_1, ζ_1 of K_1, with conjugates η_2, ζ_2; η_3, ζ_3 such that for each pair of rational integers m, n the transformation

$$\eta_j^m\zeta_j^nL_j(x,y,z)=L_j(x',y',z') \quad (j=1,2,3)$$

is an integral unimodular transformation from x,y,z to x',y',z'. Thus if the three forms take simultaneously the values $L_j=\xi_j$ for some integers x,y,z, they also take simultaneously the values $L_j=\eta_j^m\zeta_j^n\xi_j$. To make f^* small, as we shall wish to do, we shall consider numbers of this form with suitably chosen m, n. Replacing η_j, ζ_j by η_j^2, ζ_j^2 if necessary, we may assume

$$\eta_j>0, \quad \zeta_j>0 \quad (j=1,2,3); \qquad \eta_1\eta_2\eta_3=\zeta_1\zeta_2\zeta_3=1.$$

We now recast lemma 1 in a more convenient form.

LEMMA 2. *To every $\omega > 0$, however small, there is a $D = D(\omega, \eta, \zeta)$ with the following property: If ψ is given, $0 < \psi < 1$, then there are integers m, n depending on ω and ψ, such that*

$$\theta_j = \eta_j^m \zeta_j^n \quad (j = 1, 2, 3)$$

satisfies simultaneously

$$|\theta_1 - \psi\theta_2| < \omega\theta_1,$$

$$\psi^{\frac{1}{2}}\theta_i < D\theta_j \quad [i, j = 1, 2, 3;\ (i, j) \neq (2, 1)].$$

We apply lemma 1 to

$$\alpha m + \beta n = \ln \theta_1 \theta_2^{-1} = m \ln \eta_1 \eta_2^{-1} + n \ln \zeta_1 \zeta_2^{-1},$$

$$\gamma m + \delta n = \ln \theta_1 \theta_2 \quad = m \ln \eta_1 \eta_2 + n \ln \zeta_1 \zeta_2.$$

We note that α/β is irrational; for otherwise we could choose integers $(m, n) \neq (0, 0)$ such that $\alpha m + \beta n = 0$; that is, $\theta_1 = \theta_2$. But now θ_1, being equal to its conjugate, must be rational, and, being a unit, must be 1; and this contradicts the original assumption that η, ζ were independent units.

If in lemma 1 we now take (assuming $\omega < 1$)

$$\lambda = \ln \psi, \quad \tau = \ln(1 + \omega),$$

we obtain immediately

$$1 - \omega < (1 + \omega)^{-1} < \psi\theta_2\theta_1^{-1} < 1 + \omega, \quad c^{-1} \leqslant \theta_1\theta_2 \leqslant c,$$

where $c = c(\omega, \eta, \zeta) = \exp \sigma$ is independent of ψ. But these give

$$c^{-\frac{1}{2}}(1 + \omega)^{-\frac{1}{2}}\psi^{\frac{1}{2}} < \theta_1 < c^{\frac{1}{2}}(1 + \omega)^{\frac{1}{2}}\psi^{\frac{1}{2}},$$

$$c^{-\frac{1}{2}}(1 + \omega)^{-\frac{1}{2}}\psi^{-\frac{1}{2}} < \theta_2 < c^{\frac{1}{2}}(1 + \omega)^{\frac{1}{2}}\psi^{-\frac{1}{2}},$$

$$c^{-1} \leqslant \theta_3 \leqslant c,$$

since $\theta_1\theta_2\theta_3 = 1$. As c is independent of ψ, this implies the truth of the lemma.

COROLLARY. *There are also θ_j satisfying*

$$\omega\theta_1 < |\theta_1 - \psi\theta_2| < 2\omega\theta_1,$$

$$\psi^{\frac{1}{2}}\theta_i < D\theta_j \quad [i, j = 1, 2, 3;\ (i, j) \neq (2, 1)],$$

for some $D = D(\omega, \eta, \zeta)$.

This follows at once by putting

$$2\psi/(2 - 3\omega), \quad \omega/(2 - 3\omega)$$

for ψ, ω in the lemma and making a corresponding change in the value of D.

3. PROOF OF THEOREM 2

We now prove theorem 2. We first remark that if f^* takes some value δ_0 for x_0, y_0, z_0, then it takes also the values $m^3\delta_0$ $(m = \pm 1, \pm 2, \pm 3, \ldots)$ for (mx_0, my_0, mz_0). Hence it is enough to show that, given $\delta > 0$, the inequality

$$0 < |f^*| < \delta$$

is soluble for all f^* in some neighbourhood of f other than multiples of f itself.

We may suppose without loss of generality that $\epsilon_0 = 0$. We shall suppose further that

$$\epsilon_{12} = \max |\epsilon_{ij}| > 0,$$

this representing one of twelve possible cases, all of which can be treated in the same way.

We wish to find values of L_1, L_2, L_3 for which

$$0 < |L_1^* L_2^* L_3^*| < \delta.$$

To do this we shall take $L_j = \eta_j^m \zeta_j^n \xi_j$ with fixed ξ_j and choose m, n so that L_2^*, L_3^* are roughly equal to L_2, L_3, while L_1^* is much smaller than L_1. We take for ξ_1, ξ_2, ξ_3 any set of values (fixed in all that follows) taken by the L_j such that

$$\xi_1 \xi_2 < 0,$$

and put

$$\psi = -\epsilon_{12} \xi_2 \xi_1^{-1} \quad (>0).$$

Thus, since ξ, η, ζ are now fixed, an estimate of the type $\psi < \psi^* = \psi^*(\omega)$ is equivalent to one of the type

$$\epsilon_{12} = \max |\epsilon_{ij}| < \epsilon^*(\omega).$$

Now let θ_j satisfy the conditions of lemma 2, corollary, and write $L_j = \theta_j \xi_j$. Then

$$\omega\theta_1 |\xi_1| < |\theta_1 \xi_1 + \epsilon_{12} \theta_2 \xi_2| < 2\omega\theta_1 |\xi_1|.$$

Thus we have

$$\begin{aligned} |L_1^*| &= |\theta_1 \xi_1 + \epsilon_{12}\theta_2 \xi_2 + \epsilon_{13}\theta_3 \xi_3| \\ &\geqslant |\theta_1 \xi_1 + \epsilon_{12}\theta_2\xi_2| - |\epsilon_{13}\theta_3\xi_3| \\ &> \omega\theta_1 |\xi_1| - D|\xi_1 \xi_2^{-1} \xi_3| \psi^{\frac{1}{2}} \theta_1 \\ &> 0 \end{aligned}$$

if ψ is small enough. Similarly

$$|L_1^*| < 3\omega\theta_1 |\xi_1|$$

if ψ is small enough. Similarly, but more simply,

$$0 < |L_j^*| < 2\theta_j |\xi_j| \quad (j = 2, 3)$$

for small enough ψ; and so finally

$$0 < |L_1^* L_2^* L_3^*| < 12\omega |\xi_1 \xi_2 \xi_3|.$$

Since ω is arbitrarily small, this does what is required.

4. Proof of theorem 3

The same type of argument enables us to prove theorem 3. It is easy to see that the proof of lemma 1 also gives

Lemma 3. *Let α, β, γ, δ, λ be constants such that α/β is irrational and $\alpha\delta - \beta\gamma \neq 0$; then to every $\tau > 0$ however small and $\sigma > 0$ however large we can find integers m, n with*

$$|m\alpha + n\beta - \lambda| < \tau, \quad m\gamma + n\delta < -\sigma,$$

and integers m', n' with

$$|m'\alpha + n'\beta - \lambda| < \tau, \quad m'\gamma + n'\delta > \sigma.$$

The argument by which we obtained lemma 2, corollary, from lemma 1, now gives, in the notation of lemma 2:

LEMMA 4. *To any $\psi > 0$ and any $\epsilon > 0$ however small correspond integers m, n such that*

$$\epsilon\theta_1 < |\theta_1 - \psi\theta_2| < 2\epsilon\theta_1, \quad \theta_3 < \epsilon\theta_1,$$

and also integers m, n such that

$$\epsilon\theta_1 < |\theta_1 - \psi\theta_2| < 2\epsilon\theta_1, \quad \theta_2 < \epsilon\theta_3.$$

To prove theorem 3 it is enough, as in the proof of theorem 2, to show that $L_1 L_2^* L_3^*$ takes arbitrarily small non-zero values. The case when L_1, L_2^*, L_3^* are linearly dependent is trivial, since it is readily shown that the product of any three linearly dependent linear ternary forms, at least one of which does not represent zero, takes arbitrarily small non-zero values. Hence we may suppose that L_1, L_2^*, L_3^* are linearly independent and write

$$L_2^* = a_{21} L_1 + a_{22} L_2 + a_{23} L_3,$$
$$L_3^* = a_{31} L_1 + a_{32} L_2 + a_{33} L_3,$$

with
$$a_{22} a_{33} \neq a_{23} a_{32}.$$

We have now to distinguish cases. First we suppose that $a_{22} a_{23} \neq 0$. We choose ξ_1, ξ_2, ξ_3 values taken by L_1, L_2, L_3 so that $\xi_2 \xi_3 a_{22} a_{23} < 0$, and put $L_j = \theta_j \xi_j$. We regard the a_{ij} and the ξ_j as constants. Then if we choose θ_j, as we may by lemma 4, so that

$$\epsilon\theta_2 < |a_{22}\xi_2\theta_2 + a_{23}\xi_3\theta_3| < 2\epsilon\theta_2, \quad |a_{21}\theta_1\xi_1| < \epsilon\theta_2, \quad |\theta_1\xi_1| < \epsilon\theta_2,$$

we obtain, as in the proof of theorem 2,

$$L_1 = \theta_1 \xi_1 \neq 0,$$
$$0 < |L_2^*| < 3\epsilon\theta_2,$$

for small ϵ. Further,

$$a_{23} L_3^* = (a_{32} a_{23} - a_{22} a_{33}) L_2 + (a_{23} a_{31} - a_{33} a_{21}) L_1 + a_{33} L_2^*,$$

where $a_{32} a_{23} - a_{22} a_{33} \neq 0$, by hypothesis. Hence

$$|a_{23}\theta_2^{-1} L_3^*| \geq |a_{32}a_{23} - a_{22}a_{33}| \, |\xi_2| - |a_{23}a_{31} - a_{33}a_{21}| \, \epsilon - 3\,|a_{33}|\,\epsilon$$
$$> 0,$$

if ϵ is small enough. But, trivially,

$$|L_3^*| < D_3 \theta_3$$

for some D_3 depending only on the a_{ij} and the ξ_i, since $\theta_2^{-1}\theta_3$ is bounded above and below for small ϵ, by construction. Since $\theta_1\theta_2\theta_3 = 1$ this gives

$$0 < |L_1 L_2^* L_3^*| < 3\,|\xi_1|\, D_3 \epsilon,$$

which may be made as small as we please by suitable choice of ϵ.

Renumbering if necessary, we need now only consider the case

$$L_2^* = a_{21} L_1 + a_{22} L_2,$$
$$L_3^* = a_{31} L_1 \qquad\qquad + a_{33} L_3,$$

with $a_{22}a_{33} \neq 0$. If $a_{21} = a_{31} = 0$, we have the excluded case of the theorem; thus we may take $a_{21} \neq 0$. We choose ξ_1, ξ_2, ξ_3 values taken by L_1, L_2, L_3 so that $\xi_1\xi_2a_{21}a_{22} < 0$ and put $L_j = \theta_j\xi_j$. If we choose θ_j so that

$$\epsilon\theta_2 < |a_{21}\theta_1\xi_1 + a_{22}\theta_2\xi_2| < 2\epsilon\theta_2, \quad |a_{31}\theta_1\xi_1| < \epsilon\theta_3, \quad |\theta_1\xi_1| < \epsilon\theta_3,$$

we obtain in the same way as above

$$0 \neq |L_1L_2^*L_3^*| = O(\epsilon).$$

This completes the proof of theorem 3.

5. Proof of theorem 5

The proof of theorem 5 is now immediate. If there are infinitely many inequivalent sets of forms of the type specified in the theorem, then there are infinitely many lattices of determinant at most D_0 which are admissible for $|X_1X_2X_3| < 1$; and none of these is obtainable from another by a trivial transformation $X_j \to \lambda_j X_j$. By Mahler's compactness theorem (Mahler 1945, theorem II) the set of these lattices must contain a convergent subsequence. By the hypothesis that A is false, the limit lattice of this subsequence, being itself admissible for $|X_1X_2X_3| < 1$, must correspond to a product $L_1L_2L_3$ whose coefficients are proportional to integers. But now we can approximate arbitrarily closely to this product $L_1L_2L_3$ by inequivalent products $L_1^*L_2^*L_3^*$ derived from the subsequence; and for these we have $\min'|L_1^*L_2^*L_3^*| \geq 1$. Since this is in flat contradiction with theorem 2, our original assumption must have been false; and this proves theorem 5.

6. Proof of theorem 6

We now deduce theorem 6 from theorem 2. We suppose that there are ϕ, ψ such that

$$|x(\phi x - y)(\psi x - z)| \geq \delta > 0$$

for all integers $x \neq 0, y, z$. Then clearly the lattice Λ in (X_1, X_2, X_3)-space with points

$$\left.\begin{aligned} X_1 &= x \\ X_2 &= \phi x - y \\ X_3 &= \psi x - z \end{aligned}\right\} \quad (x, y, z \text{ integers})$$

is admissible for the region

$$|X_1X_2X_3| < \delta, \quad \max(|X_2|, |X_3|) < 1.$$

Hence the lattices $\Lambda^{(n)}$ obtained from Λ by the transformation

$$X_1 \to 2^{2n}X_1, \quad X_2 \to 2^{-n}X_2, \quad X_3 \to 2^{-n}X_3$$

are admissible for the respective regions

$$|X_1X_2X_3| < \delta, \quad \max(|X_2|, |X_3|) < 2^n.$$

Thus as $n \to \infty$ we can, by Mahler's general compactness principle, pick out a subsequence of the $\Lambda^{(n)}$ tending to a lattice M, where clearly M is admissible for

$$|X_1X_2X_3| < \delta.$$

Now, theorem 2 asserts in our present language that if the product $X_1X_2X_3$, $(X_1, X_2, X_3) \in \mathsf{M}$ corresponds to a multiple of a form with integral coefficients, then

$$\min |X_1X_2X_3| < \delta, \quad (X_1, X_2, X_3) \in \mathsf{M}^*, \quad X_1X_2X_3 \neq 0$$

for all lattices M^* sufficiently close to M. Hence the limit lattice M we have obtained is admissible for $|X_1X_2X_3| < \delta$ but does not correspond to a multiple of a form with integral coefficients, i.e. we have found three forms L_1, L_2, L_3 satisfying A of theorem 4. This concludes the proof of theorem 6.

7. Proof of theorem 7

Before proving theorem 7 we restate hypothesis C in what is really a dual form.

Lemma 5. *Let statement* C *hold. Then*

$$\min_{yz\neq 0} |yz(x+y\phi+z\psi)| > 0.$$

We suppose lemma 5 is false and use a cunning device of Mahler's (1939). It is trivial that $x+y\phi+z\psi \neq 0$ for any integers x, y, z not all zero. Hence given any $\epsilon_0 > 0$, with say $0 < \epsilon_0 < 1$, we could find integers x_0, y_0, z_0 such that

$$0 < |y_0z_0(x_0+y_0\phi+z_0\psi)| = \epsilon < \epsilon_0 \tag{7·1}$$

if lemma 5 were false. We make use of the identity

$$u(x_0+y_0\phi+z_0\psi)+(v-u\phi)\,y_0+(w-u\psi)\,z_0 = x_0u+y_0v+z_0w = \text{integer} \tag{7·2}$$

if u, v, w are integers. Hence, by Minkowski's linear forms theorem, we can find integers u, v, w such that

$$|v-u\phi| \leqslant \frac{\epsilon^{\frac{1}{2}}}{|y_0|} \quad (<1), \tag{7·3}$$

$$|w-u\psi| \leqslant \frac{\epsilon^{\frac{1}{2}}}{|z_0|} \quad (<1), \tag{7·4}$$

$$\begin{aligned} |x_0u+y_0v+z_0w| &< \epsilon^{-1}|y_0z_0(x_0+y_0\phi+z_0\psi)| \\ &= 1, \end{aligned} \tag{7·5}$$

since the determinant of the three linear forms on the left-hand side of (7·3), (7·4) and (7·5) is $x_0+y_0\phi+z_0\psi$ by (7·2). From (7·5) we deduce

$$x_0u+y_0v+z_0w = 0,$$

and so, by (7·2) again,

$$\begin{aligned} |u(x_0+y_0\phi+z_0\psi)| &\leqslant |y_0(v-u\phi)|+|z_0(w-u\psi)| \\ &< 2\epsilon^{\frac{1}{2}}. \end{aligned} \tag{7·6}$$

Hence by (7·1), (7·3), (7·4), (7·6) we have

$$|u(v-u\phi)\,(w-u\psi)| \leqslant 2\epsilon^{\frac{1}{2}},$$

where $u \neq 0$ by (7·4), (7·5) and since $(u, v, w) \neq (0, 0, 0)$. Since ϵ is arbitrarily small, this contradicts statement C. This contradiction proves the lemma, whose falsity was originally assumed.

We now prove theorem 7. Suppose, first, that $\phi = \phi_1$, $\psi = \psi_1$ belong to a totally real cubic field and put $L_1 = x+y\phi_1+z\psi_1$, $L_2^* = y$, $L_3^* = z$. Then theorem 3 asserts that

$$\min |yz(x+y\phi_1+z\psi_1)| = 0,$$

the minimum being taken over integers x, y, z such that

$$yz(x+y\phi_1+z\psi_1) \neq 0.$$

Hence, by lemma 5, statement C does not hold for ϕ_1, ψ_1.

Suppose therefore that $\phi = \phi_1$, $\psi = \psi_1$, where ϕ_1, ψ_1 lie in a real cubic field with conjugate imaginary fields. Let L_2, L_3, ϕ_2, ϕ_3, ψ_2, ψ_3 denote the conjugates. Then there are conjugate units η_j $(j = 1, 2, 3)$ of infinite order such that for any n the $\eta_j^n L_j$ are derived from L_j by a unimodular transformation of the variables with integer coefficients. Further η_2^n is real only for $n = 0$, since the only real elements of K_2 are rational and the only rational units are ± 1. Hence, by the one-dimensional case of Kronecker's theorem there are integral n arbitrarily large (of either sign) such that $\eta_2^n \eta_3^{-n} = (\eta_2/\bar{\eta}_2)^n$ is arbitrarily close to any given number on the unit circle.

Since the forms L_j take the values 1 they take the values $L_j = \eta_j^n$. On solving for x, y, z in terms of L_1, L_2, L_3 we have a set of equations of the type

$$x = c_1 L_1 + d_1 L_2 + \bar{d}_1 L_3,$$
$$y = c_2 L_1 + d_2 L_2 + \bar{d}_2 L_3,$$
$$z = c_3 L_1 + d_3 L_2 + \bar{d}_3 L_3,$$

where c_1, c_2, c_3 are real. We choose n, as we may from the foregoing discussion, so that $L_1 = \eta_1^n$ is arbitrarily small and also so that $|d_2\eta_2^n + \bar{d}_2\eta_3^n|\,|\eta_2^{-n}|$ is arbitrarily small. Then clearly

$$|yzL_1| = \left|\frac{y}{L_2}\right|\left|\frac{z}{L_3}\right|$$

is arbitrarily small, as asserted.

8. Proof of theorem 4

We now turn to theorem 4. Suppose first that B holds. Then A holds (with $L_j = M_j$) unless $M_1 M_2 M_3$ were a multiple of a form with integer coefficients. But in this case theorem 3 would require that $M_2(M_2+M_3)$ is a multiple of $M_2 M_3$—that is, that M_2 is a multiple of M_3, which is absurd. Thus B implies A.

To prove that A implies B we must first put the condition in A into a more useful form. This is given by

Lemma 6. *Let L_1, L_2, L_3 be three real linear forms in x_1, x_2, x_3 of non-zero determinant, each of which represents zero only trivially. Suppose there is a transformation*

$$\mathfrak{T}: \quad x_i' = \sum_j t_{ij} x_j$$

(other than the identity) with integers t_{ij}, and constants c_1, c_2, c_3 such that

$$c_j > 0, \quad c_1 c_2 c_3 = 1$$

and
$$c_j L_j(x_1, x_2, x_3) = L_j(x_1', x_2', x_3')$$

identically. Then there is a multiple of $L_1 L_2 L_3$ with integer coefficients.

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Suppose first that c_1 is rational. Then the identity in the lemma becomes

$$L_1(t_{11}-c_1, t_{21}, t_{31}) = L_1(t_{12}, t_{22}-c_1, t_{32}) = L_1(t_{13}, t_{23}, t_{33}-c_1) = 0,$$

in which all arguments are rational. Since L_1 represents zero only trivially, we have $t_{11} = t_{22} = t_{33} = c_1$, $t_{ij} = 0$ $(i \neq j)$. It follows that $c_1 = c_2 = c_3 = 1$, and the transformation is the identical one.

Thus c_1, c_2, c_3 are all irrational. Since they are the eigenvalues of the matrix (t_{ij}) they must therefore be conjugate cubic irrationals and in particular must be distinct. Thus the L_j, which are the corresponding eigenvectors, must be multiples of conjugate linear forms in conjugate cubic fields, and this proves the lemma.

If $\mathfrak{D} = (d_{ij})$ is a 3×3 unimodular matrix and Λ is a lattice, the set of all points (X'_1, X'_2, X'_3) where $X'_i = \Sigma d_{ij} X_j$ and $(X_1, X_2, X_3) \in \Lambda$ is another lattice, of the same determinant, which we denote by $\mathfrak{D}\Lambda$. Similarly, if $\mathscr{R}$ is a point set we may define the point set $\mathfrak{D}\mathscr{R}$. We say that Λ is taken into $\mathfrak{D}\Lambda$ by the transformation $\mathfrak{D}$. Clearly

$$(\mathfrak{D}_1\mathfrak{D}_2)\,\Lambda = \mathfrak{D}_1(\mathfrak{D}_2\Lambda), \quad (\mathfrak{D}_1\mathfrak{D}_2)\,\mathscr{R} = \mathfrak{D}_1(\mathfrak{D}_2\mathscr{R}),$$

and Λ is admissible for $\mathscr{R}$ if and only if $\mathfrak{D}\Lambda$ is admissible for $\mathfrak{D}\mathscr{R}$.

We write

$$\|\mathfrak{D}\| = \max_{i\neq j}(|d_{ii}-1|, |d_{ij}|),$$

so that $\|\mathfrak{D}\| = 0$ only for the unit matrix. Mahler's basic theorem on the compactness of lattices may now be put in the form (Mahler 1946), theorem 2:

Lemma 7. *Suppose that there is given an infinite set of lattices whose determinants have a common upper bound all of which are admissible for some star body* $\mathscr{R}$. *Then given* $\epsilon > 0$ *we can find two of them* $\Lambda^{(1)}, \Lambda^{(2)}$ *and a matrix* $\mathfrak{D}$ *such that*

$$\Lambda^{(2)} = \mathfrak{D}\Lambda^{(1)}, \quad \|\mathfrak{D}\| < \epsilon.$$

We shall say that $\mathfrak{D}$ is an **automorph** of Λ or $\mathscr{R}$ if $\mathfrak{D}\Lambda = \Lambda$ or $\mathfrak{D}\mathscr{R} = \mathscr{R}$ respectively.

Definition. *We shall say that a real matrix* $\mathfrak{D}$ *of determinant* 1 *is a* ***transformer to determinant*** Δ *of a region* $\mathscr{R}$ *if there is a lattice* Λ *of determinant* Δ *such that both* Λ *and* $\mathfrak{D}\Lambda$ *are admissible for* $\mathscr{R}$.

We have at once

Lemma 8. *If* $\mathfrak{S}_1, \mathfrak{S}_2$ *are automorphs of* $\mathscr{R}$ *and* $\mathfrak{D}$ *is a transformer to determinant* Δ *of* $\mathscr{R}$ *then so is* $\mathfrak{S}_1\mathfrak{D}\mathfrak{S}_2$.

For if Λ and $\mathfrak{D}\Lambda$ are admissible for $\mathscr{R}$, then $\Lambda_1 = \mathfrak{S}_2^{-1}\Lambda$ and $(\mathfrak{S}_1\mathfrak{D}\mathfrak{S}_2)\,\Lambda_1 = \mathfrak{S}_1(\mathfrak{D}\Lambda)$ are admissible for $\mathfrak{S}_2^{-1}\mathscr{R} = \mathscr{R}$ and $\mathfrak{S}_1\mathscr{R} = \mathscr{R}$ respectively.

Lemma 9. *Let* $\mathfrak{D}^{(k)} = (d_{ij}^{(k)})$ $(k = 1, 2, 3, \ldots)$ *be a sequence of transformers to determinant* Δ *for an (open) star body* $\mathscr{R}$, *and let*

$$(d_{ij}) = \mathfrak{D} = \lim_{k\to\infty} \mathfrak{D}^{(k)}$$

exist in the sense that

$$d_{ij} = \lim_{k\to\infty} d_{ij}^{(k)}.$$

Then $\mathfrak{D}$ *is a transformer to determinant* Δ *of* $\mathscr{R}$.

In the first place, $\mathfrak{D}$ is unimodular since the $\mathfrak{D}^{(k)}$ are. Let $\Lambda^{(k)}$ be a lattice of determinant Δ such that both $\Lambda^{(k)}$ and $\mathfrak{D}^{(k)}\Lambda^{(k)}$ are admissible for $\mathscr{R}$. By Mahler's compactness theorem for lattices (Mahler 1945, theorem 2) we may extract a subsequence, also to be denoted by $\Lambda^{(k)}$, which tends to a limiting lattice Λ of determinant Δ. Clearly $\mathfrak{D}\Lambda = \lim \mathfrak{D}^{(k)}\Lambda^{(k)}$. Finally, both Λ and $\mathfrak{D}\Lambda$ are admissible for $\mathscr{R}$ since $\mathscr{R}$ is open and they are the limits of lattices $\Lambda^{(k)}$, $\mathfrak{D}^{(k)}\Lambda^{(k)}$ admissible for $\mathscr{R}$ (cf. Mahler 1945, proofs of theorems 8, 9).

In view of lemma 3 the proposition 'A implies B' of theorem 4 will follow at once from the following assertion about lattices in three dimensions.

Lemma 10. *Suppose there is a lattice Λ of determinant Δ admissible for the region*

$$\mathscr{R}: \quad |X_1X_2X_3|<1$$

which has no automorphs of the type

$$X_i \to c_iX_i, \quad c_i>0, \quad c_1c_2c_3=1$$

other than the trivial $c_1=c_2=c_3=1$ (all these transformations being automorphs of $\mathscr{R}$). Then $\mathscr{R}$ has a transformer

$$\mathfrak{D}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

to determinant Δ.

For if Λ_0 (say) has transformer $\mathfrak{D}_0$ then any point (X_1, X_2, X_3) of Λ_0 other than the origin satisfies $|X_1X_2X_3|\geq 1$, since Λ_0 is admissible and $|X_1X_2(X_2+X_3)|\geq 1$ since $\mathfrak{D}_0\Lambda_0$ is admissible, i.e. B is true.

The proof of lemma 10 is now almost immediate. If Λ is given with determinant Δ we consider the lattices $\Lambda(n_1, n_2, n_3)$ derived from Λ by the transformation

$$X_i \to 2^{n_i}X_i \quad (n_i \text{ integral}, n_1+n_2+n_3=0),$$

which is an automorph for $\mathscr{R}$. Hence, by lemma 7, given $\epsilon>0$ however small, there are two of these, say

$$\Lambda_k = \Lambda(n_1^{(k)}, n_2^{(k)}, n_3^{(k)}) \quad (k=1,2),$$

and a transformer $\mathfrak{D}=(d_{ij})$ such that

$$\Lambda_2 = \mathfrak{D}\Lambda_1, \quad \|\mathfrak{D}\|<\epsilon.$$

If $\mathfrak{D}$ were a purely diagonal matrix the lattice Λ would have as automorph $X_i \to c_iX_i$, $c_i = 2^{n_i^{(1)}-n_i^{(2)}}d_{ii}$, where at least one of the c_i is not 1 if $\epsilon<\frac{1}{2}$; since then $\frac{1}{2}<d_{ii}<\frac{3}{2}$ and $(n_1^{(1)}, n_2^{(1)}, n_3^{(1)}) \neq (n_1^{(2)}, n_2^{(2)}, n_3^{(2)})$. Hence $\max|d_{ij}|>0$ $(i\neq i)$ and one particular pair i, j must give this maximum for arbitrarily small ϵ. Hence, without loss of generality we may suppose that there are transformers $\mathfrak{D}$ with $\|\mathfrak{D}\|<\epsilon$ arbitrarily small and

$$|d_{32}| = \max_{i\neq j}|d_{ij}|>0.$$

But now, by lemma 8, $\mathfrak{D}^* = \mathfrak{S}^{-1}\mathfrak{D}\mathfrak{S}$ is a transformer, where

$$\mathfrak{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{sgn} d_{32}\,|d_{32}|^{-\frac{1}{2}} & 0 \\ 0 & 0 & |d_{32}|^{\frac{1}{2}} \end{pmatrix},$$

and it is easily verified that

$$d^*_{ii} = 1+O(\epsilon),$$
$$d^*_{32} = 1,$$
$$d^*_{ij} = O(\epsilon^{\frac{1}{2}}) \quad \text{otherwise.}$$

Hence, finally, by lemma 9

$$\mathfrak{D}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

is a transformer, as asserted.

9. Statement and proof of theorem 8

In this section we shall prove the following analogue of theorem 2:

THEOREM 8. *Let $f(x,y,z)$ be a non-singular indefinite ternary quadratic form with integer coefficients, and let (δ_1, δ_2) be any open interval however small. Then there is a neighbourhood of f such that all forms f^* in the neighbourhood which are not multiples of f itself take some value in the interval (δ_1, δ_2).*

The reader will observe that we do not require that f should represent zero only trivially; thus the theorem will isolate x^2+yz as well as $x^2-3y^2-3z^2$. As in the proof of theorem 2, we note that if f^* takes some value δ_0 it takes all the values $m^2\delta_0$ $(m = 1, 2, \ldots)$, and hence it is enough to prove that f^* takes a value satisfying

$$0<f^*<\delta,$$

and a value satisfying

$$0>f^*>-\delta,$$

for any preassigned δ however small, where the neighbourhood in which f^* lies depends on δ.

No new point of principle is involved but the proof is more complicated, largely because the group of automorphisms is less convenient to handle. We write

$$f(x_1, x_2, x_3) = \Sigma f_{ij} x_i x_j.$$

If

$$g(x_1, x_2, x_3) = \Sigma g_{ij} x_i x_j$$

is another ternary quadratic form we say that f, g are **orthogonal** if

$$\sum_{i,j} f_{ij} g_{ij} = 0,$$

and we call

$$\Sigma g_{ij}^2$$

the **size** of g. Thus any ternary quadratic form f^* which is not a multiple of f can be expressed uniquely in the shape

$$f^* = (1+\epsilon_1)(f+\epsilon_2 g) \quad (\epsilon_2>0),$$

where g is orthogonal to f and has size 1. Further, a neighbourhood of f corresponds to bounds on ϵ_1, ϵ_2. As before, we can for convenience take $\epsilon_1 = 0$ and consider only

$$f^* = f+\epsilon_2 g.$$

Clearly the definitions of orthogonality and size depend on the choice of the co-ordinate system, which however we regard as fixed.† To prove the theorem, it is enough to find an automorph‡ $\mathfrak{U}$ of the form f such that the coefficient of x_1^2 in

$$\mathfrak{U}(f+\epsilon_2 g),$$

in an obvious notation, can be made less than any $\delta>0$ and of prescribed sign, provided that ϵ_2 is less than some constant depending only on f and δ.

It is now necessary to discuss the automorphs of f. If λ is an eigenvalue of an automorph $\mathfrak{T}$ then there is some linear form ξ in the variables x_1, x_2, x_3 which becomes multiplied by λ when these variables are transformed by $\mathfrak{T}$. This we shall call an **eigenform**. Suppose $\mathfrak{T}$ has distinct real eigenvalues $\lambda_1, \lambda_2, \lambda_3$, so that the corresponding eigenforms ξ_1, ξ_2, ξ_3 are uniquely determined except for scalar factors. If $f=\Sigma\alpha_{ij}\xi_i\xi_j$, then clearly $\lambda_i\lambda_j=1$ whenever $\alpha_{ij}\neq 0$. Hence one of the λ_i is ± 1 and the other two are reciprocals one of the other. Thus by taking appropriate multiples of ξ_1, ξ_2, ξ_3 and by taking $\mathfrak{T}^2$ for $\mathfrak{T}$ if need be, we have

$$f(x_1,x_2,x_3)=\rho\zeta^2+\sigma\xi\eta \quad (\rho\sigma\neq 0),$$

for some linear forms ξ, η, ζ and constants ρ, σ. Here $\mathfrak{T}$ corresponds to

$$\zeta\to\zeta, \quad \xi\to\lambda\xi, \quad \eta\to\lambda^{-1}\eta,$$

for some constant $\lambda>1$ which is irrational and lies in some quadratic field. Infinitely many quadratic fields K do occur in this way for given f, for example, all fields $\sqrt{D}$, where D is a positive integer (not a perfect square) such that $-D$ is representable by the adjoint of f.

We now choose once and for all six such automorphs $\mathfrak{T}_0, \mathfrak{T}_1, \ldots, \mathfrak{T}_5$ of f, with eigenforms§ ξ_j, η_j, ζ_j $(0\leqslant j\leqslant 5)$ and distinct quadratic fields K_j. By a suitable preliminary change of co-ordinates we may suppose that

$$(f_{11}=)\quad a_1=f(1,0,0)>0>f(0,1,0)=a_2 \quad (=f_{22}),$$

and
$$\xi_0(1,0,0)=\alpha_1\neq 0, \quad \xi_0(0,1,0)=\alpha_2\neq 0.$$

Lemma 11. *There is a constant $c_1>0$ depending only on f with the following property. To any g orthogonal to f and of size* 1 *there is an $i=i(g)$ $(1\leqslant i\leqslant 5)$, such that when g is expressed in ξ_i, η_i, ζ_i co-ordinates the coefficient of ξ_i^2 is at least c_1 in absolute value.*

The g of size 1 orthogonal to f form a closed compact set in the obvious topology, and the largest among the absolute values of the coefficients of ξ_i^2 is a continuous function $\phi(g)$ of g. It is therefore enough to prove that $\phi(g)\neq 0$ for all g. But $\phi(g)=0$ means that $g=0$ passes through the five points $\eta_i=\zeta_i=0$ $(1\leqslant i\leqslant 5)$, and these points are distinct since they lie in distinct quadratic fields. Hence $\phi(g)=0$ means that $g=0$ has five points in common with $f=0$, or that f is a multiple of g. This is clearly impossible.

† After a preliminary transformation to put f in a suitable shape to be discussed later.

‡ The term automorph, as applied to a quadratic form f, has its classical meaning, namely, an integral unimodular transformation of the variables x_1, x_2, x_3 taking f into itself. If f is indefinite it can be written (in infinitely many ways) in the standard form $\pm f=L_1^2-L_2L_3$ for some linear forms L_1, L_2, L_3. The set of values of L_1, L_2, L_3 as x_1, x_2, x_3 take integer values is a three-dimensional lattice Λ; an automorph of the form corresponds to an automorph of Λ and vice versa in the natural way.

§ Of course the new ξ_1, ξ_2, ξ_3 must not be confused with those of the earlier discussion, which will not reappear.

LEMMA 12. *If ξ_i, η_i, ζ_i are expressed in terms of ξ_0, η_0, ζ_0 in the shape*

$$\xi_i = \gamma_i \xi_0 + \text{etc.},$$
$$\eta_i = \ldots,$$
$$\zeta_i = \ldots,$$

then $\gamma_i \neq 0$.

The point $\eta_0 = \zeta_0 = 0$ lies in K_0 but is not rational; for if it were rational we should also have $\xi_0 = 0$, since ξ_0, η_0 are formally conjugate (as linear forms in x, y, z) in K_0. Hence $\eta_0 = \zeta_0 = 0$ cannot imply $\xi_i = 0$, since K_i is distinct from K_0.

LEMMA 13. *To any given $\delta_1 > 0$, $m_0 > 0$, $n_0 > 0$ there is a $\psi_0 = \psi_0(\delta_1, m_0, n_0)$ which depends only on δ_1, m_0, n_0 and the six fields $K_0, K_1, \ldots, K_5$ with the following property. To any $i = 1, 2, \ldots, 5$ and any ψ satisfying $0 < \psi < \psi_0$ there can be found integers $m > m_0$, $n > n_0$ such that*

$$1 + \delta_1 < \psi \lambda_0^{2m} \lambda_i^{2n} < 1 + 2\delta_1,$$

or again such that

$$1 - 2\delta_1 < \psi \lambda_0^{2m} \lambda_i^{2n} < 1 - \delta_1.$$

The ratio $\ln \lambda_i / \ln \lambda_1$ is irrational since λ_i, λ_1 are irrationals in distinct quadratic fields. Hence Kronecker's theorem applies, as in the proof of lemma 2, corollary.

We may now proceed to the proof of the theorem. We denote by c a constant depending only on the coefficients of the form f, and the transformations $\mathfrak{T}_i$, not necessarily the same in different contexts.

We first choose the index i by lemma 11 such that the coefficient β of ξ_i^2 in the expression for g in ξ_i, η_i, ζ_i co-ordinates satisfies

$$|\beta| \geqslant c_1 > 0.$$

By interchanging the roles of x_1, x_2 and writing $-f$ for f if need be, which does not affect our preliminary normalization, we may suppose that

$$\epsilon_2 \beta < 0.$$

We propose now to find an automorph of the type

$$\mathfrak{U} = \mathfrak{T}_0^m \mathfrak{T}_i^n \quad (m > 0,\ n > 0 \text{ integers})$$

such that the coefficient of x_1^2 in

$$\mathfrak{U}(f + \epsilon_2 g)$$

is arbitrarily small and of arbitrary sign, provided that ϵ_2 is initially small enough.

In the first place, the coefficients of ξ_i^2 in

$$\mathfrak{T}_i^n g$$

in ξ_i, η_i, ζ_i co-ordinates differs from $\beta \lambda_i^{2n}$ by at most $c\lambda_i^n$, and the remaining coefficients are at most $c\lambda_i^n$. Hence on expressing $\mathfrak{T}_i^n g$ in ξ_0, η_0, ζ_0 co-ordinates the coefficient of ξ_0^2 differs from $\beta \gamma_i^2 \lambda_i^{2n}$ ($\gamma_i \neq 0$ from lemma 12) by at most $c\lambda_i^n$; and the remaining coefficients are at most $c\lambda_i^{2n}$. Hence the coefficient of ξ_0^2 in

$$\mathfrak{T}_0^m \mathfrak{T}_i^n g$$

in ξ_0, η_0, ζ_0 co-ordinates differs from

$$\beta \gamma_i^2 \lambda_0^{2m} \lambda_i^{2n}$$

by at most $c\lambda_0^{2m} \lambda_i^n$, and the remaining coefficients are at most $c\lambda_0^m \lambda_i^{2n}$. Finally, the coefficient of x_1^2 in

$$\mathfrak{T}_0^m \mathfrak{T}_i^n g$$

expressed in x_1, x_2, x_3 co-ordinates differs from

$$\beta\gamma_i^2\alpha_i^2\lambda_0^{2m}\lambda_i^{2n}$$

by at most

$$c_0\lambda_0^m\lambda_i^{2n}+c_0\lambda_0^{2m}\lambda_i^n,$$

where $\alpha_1=\xi_0(1,0,0)\neq 0$ by hypothesis, for some constant c_0 independent of m and n. Hence finally

$$f+\epsilon_2 g$$

takes a value differing from

$$a_1+\epsilon_2\beta\gamma_i^2\alpha_i^2\lambda_0^{2m}\lambda_i^{2n}$$

by at most

$$\epsilon_2(c_0\lambda_0^m\lambda_i^{2n}+c_0\lambda_0^{2m}\lambda_i^n).$$

Let $\delta>0$ now be given arbitrarily small, and put

$$\delta=a_1\delta_1,$$

where, without loss of generality, we may suppose that

$$\delta_1<\tfrac{1}{4}.$$

We recollect that

$$a_1>0>\epsilon_2\beta.$$

First choose m_0, n_0 so large that

$$\left.\begin{aligned} c_0\lambda_0^{-m_0}&<\tfrac{1}{4}\beta\gamma_i^2\alpha_2^2\delta_1,\\ c_0\lambda_i^{n_0}&<\tfrac{1}{4}\beta\gamma_i^2\alpha_i^2\delta_1.\end{aligned}\right\}$$

Now, by lemma 13 with

$$a_1\psi=-\epsilon_2\beta\gamma_i^2\alpha_i^2 \quad (>0),$$

we can find integers $m>m_0$, $n>n_0$ so that

$$\delta<a_1+\epsilon_2\beta\gamma_i^2\alpha_i^2\lambda_0^{2m}\lambda_i^{2n}<2\delta,$$

or, again,

$$-2\delta<a_1+\epsilon_2\beta\gamma_i^2\alpha_i^2\lambda_0^{2m}\lambda_i^{2n}<-\delta,$$

provided that ϵ_2 is small enough. In both cases we have

$$|\epsilon_2\beta\gamma_i^2\alpha_i^2\lambda_0^{2m}\lambda_i^{2n}|<a_1+2\delta=a_1(1+2\delta_1),$$

and so the difference between $a_1+\epsilon_2\beta\gamma_i^2\alpha_i^2\lambda_0^{2m}\lambda_i^{2n}$ and the coefficient of x_1^2 is at most

$$\begin{aligned} c_0\lambda_0^{2m}\lambda_i^n+c_0\lambda_0^m\lambda_i^{2n}&<a_1(1+2\delta_1)\,(\tfrac{1}{4}\delta_1+\tfrac{1}{4}\delta_1)\\ &<\tfrac{3}{4}a_1\delta_1=\tfrac{3}{4}\delta,\end{aligned}$$

since initially $\delta_1<\frac{1}{4}$. Hence, finally, the coefficient b of x_1^2 satisfies either of the two equations

$$\tfrac{1}{4}\delta<b<\tfrac{19}{4}\delta$$

and

$$\tfrac{1}{4}\delta<-b<\tfrac{19}{4}\delta$$

respectively, for appropriate choice of m, n. This is what was requiring to be proved; since δ is arbitrarily small.

10. Statement and proof of theorem 9

In this section we prove the following theorem:

Theorem 9. *The following two statements are equivalent:*

D. *There is an indefinite ternary quadratic form which is not a multiple of a form with integral coefficients but*†

$$\min'|f(x,y,z)|=1.$$

E. *There are ternary linear forms M_1, M_2, M_3 such that*

$$\min'\{\min|M_2^2-M_1M_3|,\ |M_2^2-M_3(M_1+M_3)|\}=1.$$

† We recall that min′ indicates the minimum over integers x, y, z not all zero.

Since f can be put in the shape

$$\pm f = L_2^2 - L_1 L_3,$$

with ternary linear forms, an alternative form of D is

D′. *There are ternary linear forms such that* $L_2^2 - L_1 L_3$ *is not a multiple of a form with integral coefficients but*

$$\min' |L_2^2 - L_1 L_3| = 1.$$

We then have

SUPPLEMENT TO THEOREM 9. *If* D, E *are true then the lower bound of* $\det(L_1, L_2, L_3)$ *equals the lower bound of* $\det(M_1, M_2, M_3)$.

We first show that E implies D (or D′). Suppose that E is true but D is false, so that both

$$f_1(x, y, z) = M_2^2 - M_1 M_3$$

and

$$g_1(x, y, z) = M_2^2 - M_3(M_1 + M_3)$$

are multiples of integral forms f, g say. Then there are constants λ, μ such that

$$f - \lambda g = \mu L_3^2.$$

Hence λ is a double root of $\det(\mathfrak{F} - \lambda\mathfrak{G}) = 0$, where $\mathfrak{F}$, $\mathfrak{G}$ are the matrices associated with f, g; and so λ is rational. Further, M_3, being an eigenform of $\mathfrak{F} - \lambda\mathfrak{G}$, is a multiple of a rational form N, say. We may suppose that the coefficients of N are integers without common factor and so, after a suitable change of co-ordinates, that $N = z$. Hence

$$f(x, y, z) = N_2^2 - N_1 z,$$

where N_1, N_2 are multiples of M_1, M_2 respectively and need not have rational coefficients. However

$$\{N_2(x, y, 0)\}^2 = f(x, y, 0)$$

has integral coefficients, and so if

$$N_2(x, y, z) = \alpha x + \beta y + \gamma z$$

the ratio $\alpha : \beta$ is rational. Thus finally there are integers a, b such that

$$f(a, b, 0) = \{N_2(a, b, 0)\}^2 = 0.$$

Hence f, and so $f_1 = M_2^2 - M_1 M_3$, represent 0 contrary to the hypothesis that E holds.

In the rest of this section we shall show that D implies E.

LEMMA 14. *Let* $f(x_1, x_2, x_3)$ *be an indefinite quadratic form not representing* 0. *Let there exist two non-commuting automorphs of the type*

$$\mathfrak{T}: \quad x_j' = \sum_k t_{jk} x_k, \quad \det(t_{jk}) = \pm 1$$

with integral t_{jk} *each of which has three distinct eigenvalues. Then* $f(x_1, x_2, x_3)$ *is a multiple of a form with integer coefficients.*

Let $\mathfrak{T}$ be one of the automorphs with eigenvalues λ_1, λ_2, λ_3, and with eigenforms ξ_1, ξ_2, ξ_3. As in the preceding paragraph we may assume that the roots are $\lambda_1 = \lambda$, $\lambda_2 = \pm 1$, $\lambda_3 = \lambda^{-1}$. Since λ, λ^{-1} are both roots of $\det(\mathfrak{T} - \lambda\mathfrak{I}) = 0$ ($\mathfrak{I}$=unit matrix) they are algebraic units and so lie in some quadratic field since λ_1, λ_2, λ_3 are distinct. We may thus suppose without loss of generality that ξ_2, being an eigenform of $\mathfrak{T} - \lambda_2\mathfrak{I}$, has rational coefficients and that the coefficients of ξ_1, ξ_3 lie in a quadratic field and are conjugates, so that $\xi_1\xi_3$ has rational co-ordinates. Hence, as in § 9, we have

$$f(x_1, x_2, x_3) = \rho\xi_2^2 + \sigma\xi_1\xi_3,$$

where ρ, σ are real numbers and ξ_2^2 and $\xi_1\xi_3$ have rational coefficients. Here $\mathfrak{T}$ makes the transformation

$$\xi_j \to \lambda_j \xi_j.$$

If $\mathfrak{T}^*$ is another such transformation with distinct eigenvalues λ_1^*, $\lambda_2^* = \pm 1$, $\lambda_3^* = \lambda_1^{*-1}$ and eigenforms ξ_1^*, ξ_2^*, ξ_3^* it is clear, and well known, that $\mathfrak{T}^*$ commutes with $\mathfrak{T}$ if and only if the ξ_j^* are multiples of the ξ_j in some permutation. Hence under the hypothesis of the lemma we have

$$f(x_1, x_2, x_3) = \rho\xi_2^2 + \sigma\xi_1\xi_3 = \rho^*\xi_2^{*2} + \sigma^*\xi_1^*\xi_3^*,$$

where in particular ξ_2^* is not a multiple of ξ_2; and both are forms with rational coefficients. Hence we may choose a rational unimodular transformation $x_j' = \Sigma s_{jk} x_k$ such that ξ_2, ξ_2^* are multiples of x_1', x_2' respectively. After a suitable co-ordinate change we may thus assume that

$$f(x_1, x_2, x_3) = \rho_1 x_1^2 + \sigma\xi_1\xi_3 = \rho_1^* x_2^2 + \sigma^*\xi_1^*\xi_3,$$

where ρ_1, ρ_1^*, σ, σ^* are real non-zero numbers and $\xi_1\xi_3$, $\xi_1^*\xi_3^*$ are quadratic forms in x_1, x_2, x_3 with rational coefficients. Hence, by comparing the coefficients of x_1^2, x_2^2 on both sides, we see that

$$\rho_1/\sigma^*, \quad \rho_1^*/\sigma$$

are both rational. Since f is non-singular one of the terms x_1x_3, x_2x_3, x_3^2 must occur in f. Hence, by comparing coefficients, we see that

$$\sigma/\sigma^*$$

is rational. Thus finally ρ_1/σ is rational; and so f is a multiple of a form with integral coefficients, as asserted. This proves the lemma.

We must now discuss the translation of our problem into the language of the geometry of numbers. If L_1, L_2, L_3 are three linear forms of determinant Δ then

$$f = L_2^2 - L_1 L_3$$

is an indefinite quadratic form of determinant $\frac{1}{4}\Delta^2$. Conversely, if f is an indefinite ternary quadratic form then

$$\pm f = L_2^2 - L_1 L_3$$

for some linear forms L_1, L_2, L_3. This may happen in infinitely many ways but if

$$L_2^2 - L_1 L_3 = L_2^{*2} - L_1^* L_3^*,$$

the L_i^* are expressible as

$$L_i^* = \Sigma t_{ij} L_j \quad (t_{ij} \text{ real}),$$

where $\mathfrak{T} = (t_{ij})$ is an automorph of $X_2^2 - X_1 X_3$; and conversely.

We shall be concerned with automorphs of $X_2^2 - X_1 X_3$ of the special type

$$\mathfrak{S}: \begin{pmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta + \beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{pmatrix}$$

associated with a 2×2 unimodular matrix†

$$\mathfrak{S}': \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}, \quad \alpha\delta - \beta\gamma = \pm 1.$$

† This corresponds to the invariance of the discriminant of a quadratic form under unimodular transformation. It may be shown that all automorphs of $X_2^2 - X_1 X_3$ with determinant $+1$ are of this type, but we do not need this.

We always use the prime (′) to denote this correspondence between $\mathfrak{S}$ and $\mathfrak{S}'$. The eigenvalues of $\mathfrak{S}$ are $+1$ together with the squares of those of $\mathfrak{S}'$. In particular

$$(\mathfrak{S}_1\mathfrak{S}_2)' = \mathfrak{S}_1'\mathfrak{S}_2' \tag{10·1}$$

and

$$\mathfrak{S}_1 = \mathfrak{S}_2 \quad \text{if and only if} \quad \mathfrak{S}_1' = \pm\,\mathfrak{S}_2'. \tag{10·2}$$

In particular, $\mathfrak{S}_1$, $\mathfrak{S}_2$ commute if and only if

$$\mathfrak{S}_1'\mathfrak{S}_2' = \pm\mathfrak{S}_2'\mathfrak{S}_1'. \tag{10·3}$$

We also reintroduce the notation (§ 8)

$$\|\mathfrak{D}\| = \max_{i \neq j}\,(|\,d_{ii}-1\,|,\ |\,d_{ij}\,|), \tag{10·4}$$

and extend it to matrices $\mathfrak{S}'$ by putting

$$\|\mathfrak{S}'\| = \max\,(|\,\alpha-1\,|,\ |\,\delta-1\,|,\ |\,\beta\,|,\ |\,\gamma\,|). \tag{10·5}$$

Clearly

$$\|\mathfrak{S}\| \leqslant c\,\|\mathfrak{S}'\| \tag{10·6}$$

if $\|\mathfrak{S}'\| \leqslant 1$ (say), where c, as in future, denotes an absolute constant, not necessarily the same in all contexts. Further, if $\|\mathfrak{S}\|$ is small then one of the two values of $\mathfrak{S}'$ also clearly has small $\|\mathfrak{S}'\|$, and, indeed, with the correct choice of $\mathfrak{S}'$,

$$\|\mathfrak{S}'\| \leqslant c\,\|\mathfrak{S}\| \tag{10·7}$$

if $\|\mathfrak{S}\|$ is less than some constant; as may readily be verified. We also note the trivial inequalities

$$\|\mathfrak{D}_1\mathfrak{D}_2\| \leqslant c(\|\mathfrak{D}_1\|+\|\mathfrak{D}_2\|) \quad \text{if} \quad \|\mathfrak{D}_1\| \leqslant 1, \quad \|\mathfrak{D}_2\| \leqslant 1 \quad \text{(say)}, \tag{10·8$_1$}$$

and

$$\|\mathfrak{D}^{-1}\| \leqslant c\,\|\mathfrak{D}\| \tag{10·8$_2$}$$

provided that $\|\mathfrak{D}\|$ is less than some absolute constant.

We first translate lemma 12 into the new language.

Lemma 15. *Let Λ be an admissible lattice for the region*

$$\mathscr{S}: \quad |\,X_2^2 - X_1X_3\,| < 1$$

and suppose that Λ and $\mathscr{S}$ have two common non-commuting automorphs, each with three distinct eigenvalues. Then Λ corresponds to a multiple of an indefinite quadratic form with integral coefficients.

In view of the previous lemma, to show that D implies E and so complete the proof of theorem 9 it is enough to prove the following lemma:

Lemma 16. *Suppose that there exists a lattice Λ of determinant Δ which does not have two non-commuting automorphs of type $\mathfrak{S}$ each with three distinct real eigenvalues. Then*

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a transformer to determinant Δ.

For if Λ and $\mathscr{S}$ do not have two non-commuting automorphs then *a fortiori* they do not have two of the special type $\mathfrak{S}$.

Lemma 17. *Under the hypothesis of lemma 16 there exist transformers $\mathfrak{D}$ to determinant Δ which are not of the type $\mathfrak{S}$ but have arbitrarily small $\|\mathfrak{D}\|$.*

Suppose first the Λ has one automorph $\mathfrak{S}_1$, where

$$\mathfrak{S}_1' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By considering $\mathfrak{S}_0\Lambda$ with automorph $\mathfrak{S}_0\mathfrak{S}_1\mathfrak{S}_0^{-1}$ for suitable $\mathfrak{S}_0$ instead of Λ if need be, we may suppose without loss of generality that

$$\alpha\beta\gamma\delta \neq 0.$$

Now let $\epsilon > 0$ be arbitrarily small and let $\mathfrak{S}_2$ correspond to

$$\mathfrak{S}_2' = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

By lemma 7 and $(10{\cdot}8_2)$ there are integers n, m with $n+m > m \geq 0$ and a matrix $\mathfrak{D}$ with $\|\mathfrak{D}\| < \epsilon$ such that

$$\mathfrak{D}\mathfrak{S}_2^{m+n}\Lambda = \mathfrak{S}_2^m\Lambda.$$

Hence to prove the lemma it is enough to show that $\mathfrak{D}$ is not an $\mathfrak{S}$, since it is clearly a transformer to determinant Δ. We suppose that

$$\mathfrak{D} = \mathfrak{S}_3$$

and deduce a contradiction. By (10·7) we have

$$\mathfrak{S}_3' = \begin{pmatrix} 1+O(\epsilon) & O(\epsilon) \\ O(\epsilon) & 1+O(\epsilon) \end{pmatrix},$$

where the constant implied by the O is absolute. The lattice Λ clearly has the automorph

$$\mathfrak{S}_4 = \mathfrak{S}_2^{-m}\mathfrak{S}_3\mathfrak{S}_2^{m+n},$$

where

$$\mathfrak{S}_4' = \begin{pmatrix} 2^{-m} & 0 \\ 0 & 2^m \end{pmatrix}\begin{pmatrix} 1+O(\epsilon) & O(\epsilon) \\ O(\epsilon) & 1+O(\epsilon) \end{pmatrix}\begin{pmatrix} 2^{m+n} & 0 \\ 0 & 2^{-m-n} \end{pmatrix}$$
$$= \begin{pmatrix} 2^n\{1+O(\epsilon)\} & O(2^{-2m-n}\epsilon) \\ O(2^{2m+n}\epsilon) & 2^{-n}\{1+O(\epsilon)\} \end{pmatrix};$$

so $\mathfrak{S}_4$ has three distinct real eigenvalues if ϵ is small enough. Further, $\mathfrak{S}_1'\mathfrak{S}_4' \neq \pm\mathfrak{S}_4'\mathfrak{S}_1'$ if ϵ is smaller than some $\epsilon_0(\alpha,\beta,\gamma,\delta) > 0$, since the top right-hand elements of $\mathfrak{S}_1'\mathfrak{S}_4'$ and $\mathfrak{S}_4'\mathfrak{S}_1'$ are respectively

$$\alpha 2^{-2m-n}O(\epsilon) + 2^{-n}\beta\{1+O(\epsilon)\},$$

$$2^n\beta\{1+O(\epsilon)\} + \delta 2^{-2m-n}O(\epsilon);$$

and $\beta \neq 0$ by our preliminary transformation. Hence Λ has the two non-commuting automorphs $\mathfrak{S}_1$, $\mathfrak{S}_4$ contrary to hypothesis. Hence $\mathfrak{D}$ is not an $\mathfrak{S}$ and the lemma holds in this case.

If, however, we assume that initially Λ has no automorphs $\mathfrak{S}$, then the foregoing line of argument, omitting all reference to $\mathfrak{S}_1$, constructs an automorph $\mathfrak{S}_4$ unless there are transformers $\mathfrak{D}$ with arbitrarily small $\|\mathfrak{D}\|$. This proves the lemma.

Lemma 18. *Under the hypothesis of lemma* 16 *there are indeed transformers* $\mathfrak{D} = (d_{ij})$ *to determinant* Δ *not of the form* $\mathfrak{S}$, *with*

$$d_{12} = d_{32} = 0, \quad d_{11} = d_{33}$$

and arbitrarily small $\|\mathfrak{D}\|$.

If $\mathfrak{D}$ is a transformer with small $\|\mathfrak{D}\|$ but not an $\mathfrak{S}$ then

$$\mathfrak{D}^* = \mathfrak{S}_1\mathfrak{D}$$

will have the same properties if $\|\mathfrak{S}_1\|$ is small, since the $\mathfrak{S}$ are a group under multiplication and ($10\cdot8_1$) holds. Suppose that $\mathfrak{D}$ with $\|\mathfrak{D}\|<\epsilon$ is given by lemma 17 and try to choose $\mathfrak{S}_1$, where

$$\mathfrak{S}_1' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

so that $\mathfrak{D}^*$ satisfies the conditions of lemma 15. We have to choose $\alpha, \beta, \gamma, \delta$ so that

$$\alpha\delta-\beta\gamma = 1 \tag{10·9}$$

and

$$d^*_{12} = \alpha^2 d_{12}+2\alpha\gamma d_{22}+\gamma^2 d_{32} = 0, \tag{10·10}$$

$$d^*_{32} = \beta^2 d_{12}+2\beta\delta d_{22}+\delta^2 d_{32} = 0, \tag{10·11}$$

$$\begin{aligned} d^*_{11} &= \alpha^2 d_{11}+2\alpha\gamma d_{21}+\gamma^2 d_{31} \\ &= \beta^2 d_{13}+2\beta\delta d_{23}+\delta^2 d_{33} = d^*_{33}. \end{aligned} \tag{10·12}$$

Put $\gamma = \lambda\alpha$, $\beta = \mu\delta$. Since $|d_{22}-1|<\epsilon$, $|d_{12}|<\epsilon$, $|d_{32}|<\epsilon$ we can choose λ, μ such that†

$$|\lambda|<c\epsilon, \quad |\mu|<c\epsilon,$$

and (10·10), (10·11) are satisfied, provided that ϵ is small enough. The equations (10·9) and (10·12) now become

$$\alpha\delta(1-\lambda\mu) = 1,$$

and

$$\alpha^2(d_{11}+2\lambda d_{21}+\lambda^2 d_{31}) = \delta^2(d_{33}+2\mu d_{23}+\mu^2 d_{13}).$$

Since

$$1-\lambda\mu = 1+O(\epsilon^2) = 1+O(\epsilon),$$

$$d_{11}+2\lambda d_{21}+\lambda^2 d_{31} = 1+O(\epsilon),$$

$$d_{33}+2\mu d_{23}+\mu^2 d_{13} = 1+O(\epsilon),$$

we may clearly satisfy these equations with

$$|\alpha-1|<c\epsilon, \quad |\delta-1|<c\epsilon.$$

Hence

$$\|\mathfrak{S}_1'\|\leqslant c\epsilon,$$

and consequently

$$\|\mathfrak{S}_1\|\leqslant c\epsilon$$

and

$$\|\mathfrak{D}^*\|\leqslant c(\|\mathfrak{D}\|+\|\mathfrak{S}_1\|)\leqslant c\epsilon$$

by (10·6) and ($10\cdot8_1$). Since ϵ is arbitrarily small, this does what is required.

LEMMA 19. *If $\mathfrak{D}$ is as in lemma 18, then*

$$\max(|d_{11}+2d_{22}+d_{33}|, |d_{13}|, |d_{21}|, |d_{23}|, |d_{31}|) = \|\mathfrak{D}\|+O(\|\mathfrak{D}\|^2),$$

where the constant implied by O is absolute.

Since all the d_{ij} which are not zero ($i\neq j$) occur on the right-hand side it is enough to show that

$$|d_{12}-2d_{22}+d_{13}| = \max|d_{ii}-1|+O(\|\mathfrak{D}\|^2).$$

Put $d_{11} = d_{33} = 1+\delta_1$, $d_{22} = 1+\delta_2$. Then

$$\begin{aligned} 1 = \det\mathfrak{D} &= d_{11}d_{22}d_{33}+O(\|\mathfrak{D}\|^2) \\ &= 1+2\delta_1+\delta_2+O(\|\mathfrak{D}\|^2). \end{aligned}$$

† We remind the reader that c is an absolute constant, not necessarily the same in different contexts.

Hence
$$\delta_2 = -2\delta_1 + O(\|\mathfrak{D}\|^2);$$
and the result is immediate.

LEMMA 20. *If A_j $(0 \leqslant j \leqslant 4)$ are any five numbers and*
$$f(\beta) = A_0 + A_1\beta + \ldots + A_4\beta^4,$$
then
$$\max_j |A_j| \leqslant c \max |f(\beta)| \quad (\beta = 0, \pm 1, \pm 2).$$
for some absolute constant c.

For the A_j can be expressed in terms of the $f(\beta)$ by linear equations with constant coefficients.

LEMMA 21. *Under the hypothesis of lemma 16 there are transformers $\mathfrak{D}$ not of type $\mathfrak{S}$ with arbitrarily small $\|\mathfrak{D}\|$ and*
$$\|\mathfrak{D}\| \leqslant c\,|d_{13}|.$$

If $\mathfrak{D}$ is given by lemma 15 with $\|\mathfrak{D}\| < \epsilon$ we show that
$$\mathfrak{D}^* = \mathfrak{S}_\beta \mathfrak{D} \mathfrak{S}_\beta^{-1},$$
where
$$\mathfrak{S}'_\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$
will do what is required, for suitable $\beta = 0, \pm 1, \pm 2$. Indeed, in the first place
$$\|\mathfrak{D}^*\| \leqslant c\,\|\mathfrak{D}\| \tag{10·13}$$
for each β, since $\mathfrak{D}$ differs from the unit matrix by terms at most ϵ in absolute value. On the other hand,
$$d^*_{13} = d_{13} + 2d_{23}\beta + (d_{11} - 2d_{22} + d_{33})\,\beta^2 + 2d_{21}\beta^3 + d_{31}\beta^4,$$
and so
$$\|\mathfrak{D}\| \leqslant c\,|d^*_{13}| \tag{10·14}$$
by the two preceding lemmas if β is suitably chosen from $0, \pm 1, \pm 2$. Hence, by (10·13), (10·14), we have
$$\|\mathfrak{D}^*\| \leqslant c\,|d^*_{13}|,$$
where $\|\mathfrak{D}^*\|$ and $|d^*_{13}|$ may be arbitrarily small. We note that $\mathfrak{D}^*$ is not the unit matrix since it is not an $\mathfrak{S}$, and hence that $d_{13} \neq 0$.

The proof of lemma 16 is now almost immediate. Let $\mathfrak{D} = (d_{ij})$ be given by the last lemma and let $|d_{13}| = \epsilon$, so
$$|d_{ii} - 1| \leqslant c\epsilon, \quad |d_{ij}| \leqslant c\epsilon \quad (i \neq j).$$
Then
$$\mathfrak{D}^* = \begin{pmatrix} |d_{13}|^{-\frac{1}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |d_{13}|^{\frac{1}{2}} \end{pmatrix} \mathfrak{D} \begin{pmatrix} |d_{13}|^{\frac{1}{2}} \operatorname{sgn} d_{13} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |d_{13}|^{-\frac{1}{2}} \operatorname{sgn} d_{13} \end{pmatrix}$$
is also a transformer by lemma 5 (since the first and last factors are $\mathfrak{S}$'s). Clearly
$$|d^*_{ii} - 1| \leqslant c\epsilon, \quad d^*_{13} = 1,$$
$$|d^*_{ij}| \leqslant c\epsilon^{\frac{1}{2}} \quad \text{otherwise.}$$
Finally, the limit
$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a transformer to determinant Δ by lemma 9. This concludes the proof of lemma 16, and so of theorem 9.

11. Statement of theorem 10

Just as theorem 5 was deduced from theorem 2, so the following theorem may be deduced from theorem 8. We suppress the proof, which is virtually identical.

Theorem 10. *If statements* D, E *are false, then to any D_0 however large there are only a finite number of inequivalent indefinite ternary quadratic forms f with determinant at most D_0 such that*

$$\min' |f| = 1.$$

Appendix A

With his permission we give here Professor Littlewood's proof that

$$\lim_{n\to\infty} n^{1+\epsilon} |\sin \pi\phi n \sin \pi\psi n| = \infty$$

for all $\epsilon > 0$ and almost all ϕ, ψ.

Choose $\zeta > 0$, $\eta > 0$ so that

$$(1+\zeta)(1-\eta)^{-1} = 1+\epsilon,$$

and put

$$f(\phi,\psi) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\zeta} |\sin \pi\phi n \sin \pi\psi n|^{1-\eta}},$$

so that $0 \leqslant f(\phi,\psi) \leqslant \infty$. Then clearly

$$\iint_{\substack{0\leqslant\phi<1\\0\leqslant\psi<1}} f(\phi,\psi)\,\mathrm{d}\phi\,\mathrm{d}\psi = \left(\sum_{1}^{\infty} \frac{1}{n^{1+\zeta}}\right)\left(\iint_{\substack{0\leqslant\phi<1\\0\leqslant\psi<1}} \frac{\mathrm{d}\phi\,\mathrm{d}\psi}{|\sin \pi\phi \sin \pi\psi|^{1-\eta}}\right) < \infty;$$

and so $f(\phi,\psi) < \infty$ almost everywhere. But $f(\phi,\psi) < \infty$ implies that

$$n^{1+\zeta} |\sin \pi\phi n \sin \pi\psi n|^{1-\eta} \to \infty,$$

and so

$$n^{1+\epsilon} |\sin \pi\phi n \sin \pi\psi n| \to \infty,$$

as asserted.

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